

Supersymmetry breaking on the lattice: the $\mathcal{N} = 1$ Wess-Zumino model

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We discuss spontaneous supersymmetry breaking in the $\mathcal{N} = 1$ Wess-Zumino model in two dimensions on the lattice using Wilson fermions and the fermion loop formulation. In that formulation the fermion sign problem related to the vanishing of the Witten index can be circumvented and the model can be simulated very efficiently using the recently introduced open fermion string algorithm. We present first results for the supersymmetry breaking phase transition and sketch the preliminary determination of a renormalised critical coupling in the continuum limit.

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1. Introduction

Supersymmetry as an extension of the space-time symmetries is an interesting concept which might well be realised in nature in one form or another. Its presence has several intriguing consequences, for example, one expects a vanishing ground state energy, if supersymmetry is exact. Moreover, the particle spectrum contains mass degenerate bosons and fermions which are related by the supersymmetry. However, so far there has not been any experimental sign of such a boson-fermion degeneracy. So, if supersymmetry is indeed realised at some high energy scale, it must be broken at the low energy scales accessible in today's experiments. The scenario of (spontaneously) broken supersymmetry then implies that there is no supersymmetric ground state, the ground state energy is not vanishing, and the particle masses need not be degenerate. In this context, it is interesting to ask how the spontaneous breaking of the supersymmetry is realised. Since spontaneous symmetry breaking is an inherently non-perturbative problem one needs non-perturbative methods in order to approach it meaningfully. One such method is provided by simulating supersymmetric theories on a space-time lattice. However, since the space-time symmetries are explicitly broken by the lattice regularisation and are restored only in the continuum limit, also the supersymmetry is in general not (or not fully) realised on the lattice. As a consequence, there is a subtle and delicate interplay between the various symmetries, and their realisation in the continuum needs to be carefully studied. Here we present preliminary results of such a study for the $\mathcal{N} = 1$ Wess-Zumino model in two dimensions. Using the Wilson fermion discretisation one can formulate the model in terms of fermion loops which can be simulated very efficiently using the open fermion string algorithm [1]. In addition, the fermion loop formulation provides a way to circumvent the sign problem related to the vanishing of the Witten index [2].

2. The $\mathcal{N} = 1$ Wess-Zumino model

The $\mathcal{N} = 1$ Wess-Zumino model in two dimensions is one of the simplest models which may exhibit spontaneous supersymmetry breaking. Its degrees of freedom consist of one real Majorana fermion field ψ and one real bosonic field ϕ , while its dynamics is described by the Lagrangian density

$$\mathcal{L} = \frac{1}{2} (\partial_\mu \phi)^2 + \frac{1}{2} P'(\phi)^2 + \frac{1}{2} \bar{\psi} (\partial + P''(\phi)) \psi. \quad (2.1)$$

Here, $P(\phi)$ denotes a generic superpotential, and P', P'' its first and second derivative with respect to ϕ . In the following we will concentrate on the specific form

$$P(\phi) = \frac{m^2}{4g} \phi + \frac{1}{3} g \phi^3 \quad (2.2)$$

which leads to a vanishing Witten index $W = 0$ and hence allows for spontaneous supersymmetry breaking [3]. The corresponding action enjoys the following two symmetries. First, there is a single supersymmetry given by the transformations

$$\delta \phi = \bar{\epsilon} \psi, \quad \delta \psi = (\partial \phi - P') \epsilon, \quad \delta \bar{\psi} = 0, \quad (2.3)$$

and second, there is a discrete $\mathbb{Z}(2)$ chiral symmetry given by

$$\phi \rightarrow -\phi, \quad \psi \rightarrow \gamma_5 \psi, \quad \bar{\psi} \rightarrow -\bar{\psi} \gamma_5, \quad (2.4)$$

where $\gamma_5 \equiv \sigma_3$ can be chosen to be the third Pauli matrix.

For the chosen superpotential the Witten index turns out to be zero, as can be seen as follows. Integrating out the Majorana fermions yields the (indefinite) Pfaffian $\text{Pf}M$ of the Majorana Dirac operator M . The corresponding partition function with periodic boundary conditions (b.c.) in all directions is equivalent to the Witten index,

$$\int \mathcal{D}\phi e^{-S_b(\phi)} \text{Pf}M_{\text{pp}}(\phi) \propto W,$$

where $S_b(\phi)$ is the action for the bosonic field. Now, under the $\mathbb{Z}(2)$ symmetry $\phi \rightarrow -\phi$ one has

$$S_b \rightarrow S_b, \quad \text{Pf}M_{\text{pp}} \rightarrow -\text{Pf}M_{\text{pp}},$$

so for every bosonic field configuration ϕ contributing to the partition function, there exists another one with exactly the same contribution but opposite sign, hence yielding $W = 0$. This constitutes a necessary (but not sufficient) condition for the supersymmetry to be broken spontaneously. In that case, one expects a bosonic and fermionic ground state related to each other by the supersymmetry transformation. On the other hand, if one chooses thermal b.c. (antiperiodic b.c. for the fermions in time direction) the supersymmetry is broken by the finite temperature of the system and one finds

$$S_b \rightarrow S_b, \quad \text{Pf}M_{\text{pa}} \rightarrow +\text{Pf}M_{\text{pa}}.$$

In order to further understand the supersymmetry breaking pattern, i.e. the relation between the supersymmetry breaking and the $\mathbb{Z}(2)$ symmetry breaking, it is useful to consider the potential for the bosonic field, $\frac{1}{2}P'(\phi)^2 = \frac{1}{2}\frac{m^2}{2}\phi^2 + \frac{1}{2}g^2\phi^4 + \text{const}$. It simply represents a standard ϕ^4 -theory in which, depending on the choice of the bare parameters m and g , the $\mathbb{Z}(2)$ symmetry may be broken. Indeed, for large values of m/g the $\mathbb{Z}(2)$ symmetry is spontaneously broken (in infinite volume) and the boson field selects a definite ground state. Denoting with $\bar{\phi}$ the expectation value of the volume averaged boson field, one finds

$$\begin{aligned} \bar{\phi} = +m/2g &\Rightarrow \text{Pf}M_{\text{pp}} = +\text{Pf}M_{\text{pa}}, \\ \bar{\phi} = -m/2g &\Rightarrow \text{Pf}M_{\text{pp}} = -\text{Pf}M_{\text{pa}}, \end{aligned}$$

i.e. in the former case the unique ground state is bosonic, while in the latter it is fermionic. In both cases, there is a single, unique ground state tantamount to having unbroken supersymmetry. By contrast, for small values of m/g the $\mathbb{Z}(2)$ symmetry is unbroken, i.e. one has $\bar{\phi} = 0$ which allows both a bosonic and fermionic ground state, tantamount to having broken supersymmetry. Indeed, the tunneling between the two equivalent ground states corresponds to the massless Goldstino mode which comes along with any spontaneous supersymmetry breaking.

2.1 Lattice discretisation and fermion loop formulation

In order to put the Wess-Zumino model on the lattice we follow the approach of Golterman and Petcher [4] where it is shown that using the same lattice derivative for the bosons as for the fermions (and renormalising the mass parameter m accordingly), the supersymmetry is guaranteed to be restored in the continuum limit.

Using the Wilson lattice discretisation for the fermion fields yields the fermion Lagrangian density

$$\mathcal{L} = \frac{1}{2} \xi^T \mathcal{C} (\gamma_\mu \tilde{\partial}_\mu - \frac{1}{2} \partial^* \partial + P''(\phi)) \xi ,$$

where ξ is a real, 2-component Grassmann field, $\mathcal{C} = -\mathcal{C}^T$ is the charge conjugation matrix and ∂^*, ∂ are the backward and forward lattice derivatives, respectively. However, while the Wilson term $\partial^* \partial$ avoids fermion doubling, it spoils the discrete chiral symmetry of the fermion action as well as the $\mathbb{Z}(2)$ symmetry $\phi \rightarrow -\phi$ of the boson action¹.

Another problem for simulating the model on the lattice is the fact that the Pfaffian is indefinite. As discussed above, this is due to the vanishing of the Witten index and constitutes a generic problem for any numerical Monte Carlo investigation of spontaneous supersymmetry breaking, independent of the chosen discretisation. The problem stems from the fact that the indefinite Pfaffian can not be simulated directly by standard Monte Carlo methods. Instead one uses the effective action

$$S_{\text{eff}}(\phi) = S_b(\phi) - \ln |\text{Pf} M(\phi)|$$

for the boson field ϕ and takes the sign of the Pfaffian into account by reweighting. This approach in general leads to severe sign problems [5, 6].

It turns out that the sign problem can be circumvented for Wilson fermions by using an exact reformulation of the lattice model in terms of fermion loops [2] and simulating fluctuating fermionic boundary conditions [1]. In the loop formulation one expands the Boltzmann factor of the fermion action, effectively constructing a hopping expansion. When one subsequently performs the integration over the fermion fields, the nil-potency of the Grassmann elements ensures that only closed, non-oriented and non-intersecting fermion loops survive. The partition function then becomes a sum over all self-avoiding fermion loop configurations ℓ ,

$$Z_{\mathcal{L}} = \sum_{\{\ell\} \in \mathcal{L}} \omega[\ell, \phi], \quad \mathcal{L} \in \mathcal{L}_{00} \cup \mathcal{L}_{10} \cup \mathcal{L}_{01} \cup \mathcal{L}_{11}$$

where $\omega[\ell, \phi]$ denotes the weight for a given loop configuration ℓ , and \mathcal{L}_{ij} denotes the equivalence class of loop configurations with an even or odd number of loops winding around the lattice in the spatial and temporal direction, respectively. $Z_{\mathcal{L}}$ represents a system with unspecified fermionic b.c. [7], while the system with periodic b.c. for the fermion, i.e. the Witten index, can be constructed by forming

$$W \equiv Z_{\text{pp}} = Z_{\mathcal{L}_{00}} - Z_{\mathcal{L}_{10}} - Z_{\mathcal{L}_{01}} - Z_{\mathcal{L}_{11}} ,$$

or the system at finite temperature by forming

$$Z_{\text{pa}} = Z_{\mathcal{L}_{00}} - Z_{\mathcal{L}_{10}} + Z_{\mathcal{L}_{01}} + Z_{\mathcal{L}_{11}} .$$

Note that the weight ω does not necessarily need to be positive definite in each of the sectors, but in practice it turns out that it is the case as long as one stays close enough to the continuum limit.

As described in [1] the system can most efficiently be simulated by introducing an open fermion string corresponding to the insertion of a Majorana fermion pair. By letting the ends

¹In principle this complication can be avoided by using a lattice discretisation which respects the discrete $\mathbb{Z}(2)$ chiral symmetry, e.g. the SLAC derivative [5].

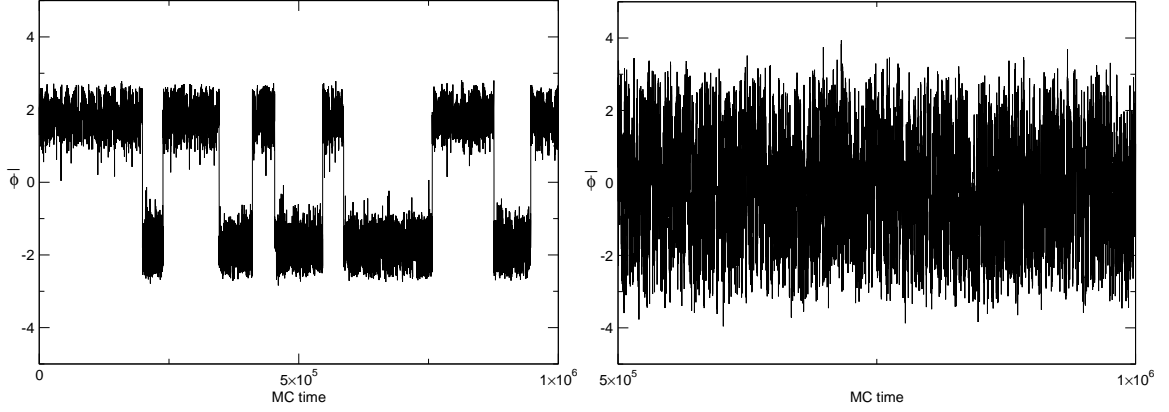


Figure 1: Monte Carlo time history of $\bar{\phi}$ at $m/g = 4$, $ag = 0.125$ (left plot) and $m/g = 0.16$, $ag = 0.03125$ (right plot).

of the string move around the lattice by a standard Metropolis update procedure, one samples the fermion 2-point function as well as the relative weights between $Z_{\mathcal{L}_{00}}, Z_{\mathcal{L}_{10}}, Z_{\mathcal{L}_{01}}$ and $Z_{\mathcal{L}_{11}}$. Finally, the bosonic fields are simulated by standard Monte Carlo methods.

3. Results

By looking at the behaviour of $\bar{\phi}$ for large and small values of m/g one can check for the $\mathbb{Z}(2)$ symmetry breaking². The plots in Fig. 1 show the Monte Carlo time history of $\bar{\phi}$ at $m/g = 4$, $ag = 0.125$ (left plot) and $m/g = 0.16$, $ag = 0.03125$ (right plot). For $m/g = 4$ the system is in the $\mathbb{Z}(2)$ broken phase with $\langle |\bar{\phi}| \rangle \simeq 2$. Since the system is at finite volume it still tunnels between the two vacua with $\langle |\bar{\phi}| \rangle \simeq \pm m/2g$, but the tunnelling will be suppressed in the limit $L \rightarrow \infty$ or $m/g \rightarrow \infty$. For $m/g = 0.16$ on the other hand, the system is in the $\mathbb{Z}(2)$ symmetric phase with $\langle \bar{\phi} \rangle \simeq 0$.

It is now interesting to see how the partition functions $Z_{\mathcal{L}_{00}}, Z_{\mathcal{L}_{10}}, Z_{\mathcal{L}_{01}}, Z_{\mathcal{L}_{11}}$, or Z_{pp} and Z_{pa} , behave in the two situations. Fig. 2 shows the probability distributions of the various sectors as a function of $\bar{\phi}$, again for $m/g = 4$, $ag = 0.125$ (left plot) and $m/g = 0.16$, $ag = 0.03125$ (right plot). In the first situation where the $\mathbb{Z}(2)$ symmetry is broken, one finds

$$\begin{aligned} \langle \bar{\phi} \rangle \simeq -2: \quad & Z_{00} \simeq Z_{10} \simeq Z_{01} \simeq Z_{11} \quad \Rightarrow \quad Z_{pp} \simeq -Z_{pa}, \\ \langle \bar{\phi} \rangle \simeq +2: \quad & Z_{00} \simeq 1, Z_{10} \simeq Z_{01} \simeq Z_{11} \simeq 0 \quad \Rightarrow \quad Z_{pp} \simeq +Z_{pa}, \end{aligned}$$

so $\langle \bar{\phi} \rangle \simeq -2$ corresponds to the fermionic ground state while $\langle \bar{\phi} \rangle \simeq +2$ corresponds to the bosonic one. In either case, a unique ground state is chosen by the system (up to finite volume tunneling) and hence supersymmetry is unbroken (at least in the thermodynamic and continuum limit).

In the second situation, where the $\mathbb{Z}(2)$ -symmetry is unbroken, one finds

$$\langle \bar{\phi} \rangle \simeq 0: \quad Z_{00} \simeq Z_{10} + Z_{01} + Z_{11} \quad \Rightarrow \quad Z_{pp} \simeq 0,$$

so the bosonic and fermionic ground states occur with equal probability (thereby cancelling their contribution in $Z_{pp} = W$) and hence supersymmetry is spontaneously broken.

As a next step one can now determine the value of m/g at which the transition from the

²Note that $\bar{\phi}$ is not a true order parameter for the $\mathbb{Z}(2)$ symmetry since the symmetry is explicitly broken by the lattice discretisation.

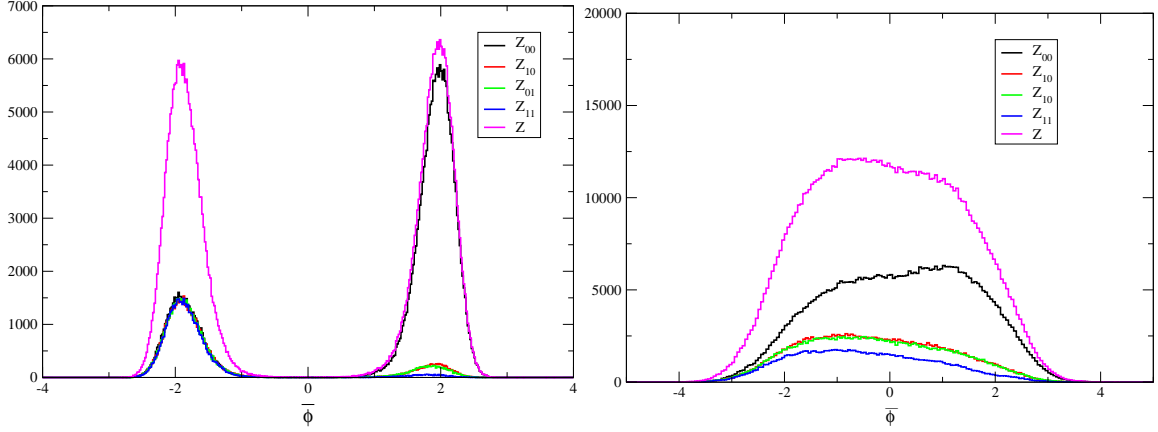


Figure 2: Probability distribution of the partition functions $Z_{\mathcal{L}_{00}}, Z_{\mathcal{L}_{10}}, Z_{\mathcal{L}_{01}}, Z_{\mathcal{L}_{11}}$ and $Z_{\mathcal{L}}$ for $m/g = 4$, $ag = 0.125$ (left plot) and $m/g = 0.16$, $ag = 0.03125$ (right plot) as a function of $\bar{\phi}$.

$\mathbb{Z}(2)$ symmetric and supersymmetry broken phase to the $\mathbb{Z}(2)$ broken and supersymmetric phase occurs. This is most easily done by scanning am at fixed lattice spacing ag for various lattice extents L . The critical value am_c where the phase transition occurs determines the dimensionless critical coupling m_c/g at the given lattice spacing. The procedure is illustrated in Fig. 3 where the left plot shows the ratio Z_{pp}/Z serving as a (pseudo)-order parameter for the supersymmetry breaking phase transition, while the right plot shows $\langle s_\phi \rangle_{pa} = \langle \text{sign} \bar{\phi} \rangle_{pa}$ as a (pseudo)-order parameter for the $\mathbb{Z}(2)$ symmetry breaking³, as a function of the bare mass at fixed lattice spacing $ag = 0.03125$ for various lattice extents. We note that the behaviour of s_ϕ seems to suggest a second order phase transition. Setting up the model with the Wilson derivative for bosons and fermions yields a supersymmetric continuum limit [4]. Since the model is superrenormalisable it is

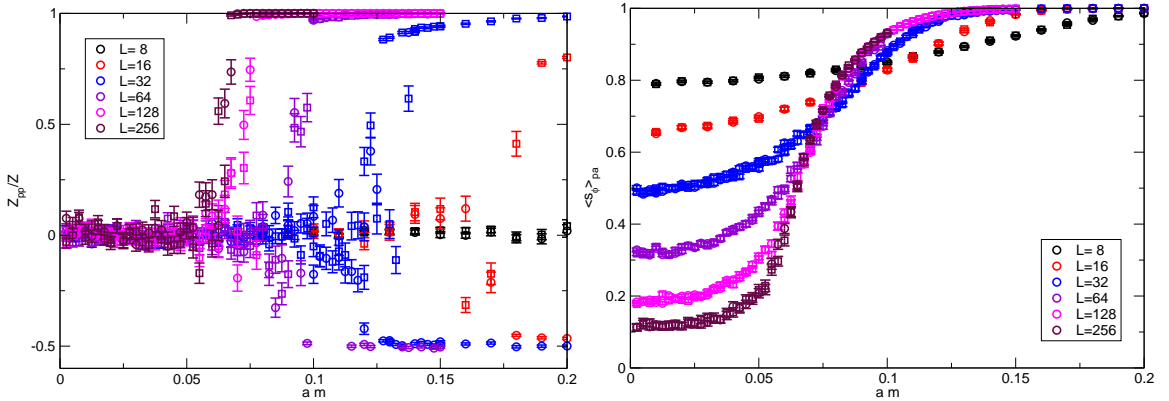


Figure 3: Ratio Z_{pp}/Z (left plot) and $\langle s_\phi \rangle = \langle \text{sign} \bar{\phi} \rangle$ serving as a (pseudo)-order parameter for the supersymmetry breaking phase transition and the $\mathbb{Z}(2)$ -symmetry breaking phase transition, respectively, as a function of am at fixed lattice spacing $ag = 0.03125$ for various lattice extents.

³Note that since $Z_{pp} \simeq 0$ in the supersymmetry broken phase, expectation values need to be calculated in the thermal ensemble in order to be under good numerical control.

sufficient to tune only the mass parameter m in order to obtain a renormalised theory. Using the relation $m^2 = m_R^2 + 2g^2/\pi \ln m_R^2$ between the bare mass m and the renormalised one m_R , and setting the scale, i.e. the lattice spacing a , by the dimensionful coupling $g = \hat{g}/a$, one can determine the continuum limit of a dimensionless critical coupling for the supersymmetry breaking transition with

$$f_{\text{crit}} = \lim_{\hat{g} \rightarrow 0} \frac{g}{m_R} \Big|_{\text{crit}}.$$

The procedure is illustrated in Fig. 4 and it will be interesting to see how this result compares to previous determinations [5, 8].

4. Outlook

There are several obvious ways to proceed from this first, preliminary investigation. Firstly, one can use Ward identities as (pseudo-)order parameters to determine the phase transition point. Secondly, one can determine the boson and fermion mass spectra. The latter is particularly simple in the fermion loop formulation. Thirdly, one can try to perform non-perturbative renormalisation using the boson or fermion masses. Finally, it would also be interesting to implement the loop formulation of the model with a domain wall or overlap type fermion discretisation for which the discrete $\mathbb{Z}(2)$ chiral symmetry remains exact at finite lattice spacing.

References

- [1] U. Wenger, *Efficient simulation of relativistic fermions via vertex models*, *Phys. Rev.* **D80** (2009) 071503, [[arXiv:0812.3565](#)].
- [2] D. Baumgartner and U. Wenger, *Simulation of supersymmetric models on the lattice without a sign problem*, [arXiv:1104.0213](#).
- [3] E. Witten, *Constraints on Supersymmetry Breaking*, *Nucl. Phys.* **B202** (1982) 253.
- [4] M. F. L. Golterman and D. N. Petcher, *A local interactive lattice model with supersymmetry*, *Nucl. Phys.* **B319** (1989) 307–341.
- [5] C. Wozar and A. Wipf, *Supersymmetry Breaking in Low Dimensional Models*, [arXiv:1107.3324](#).
- [6] S. Catterall and S. Karamov, *A lattice study of the two-dimensional Wess Zumino model*, *Phys. Rev.* **D68** (2003) 014503, [[hep-lat/0305002](#)].
- [7] U. Wolff, *Cluster simulation of relativistic fermions in two space- time dimensions*, *Nucl. Phys.* **B789** (2008) 258–276, [[arXiv:0707.2872](#)].
- [8] M. Beccaria, G. F. De Angelis, M. Campostrini, and A. Feo, *Phase diagram of the lattice Wess-Zumino model from rigorous lower bounds on the energy*, *Phys. Rev.* **D70** (2004) 035011, [[hep-lat/0405016](#)].

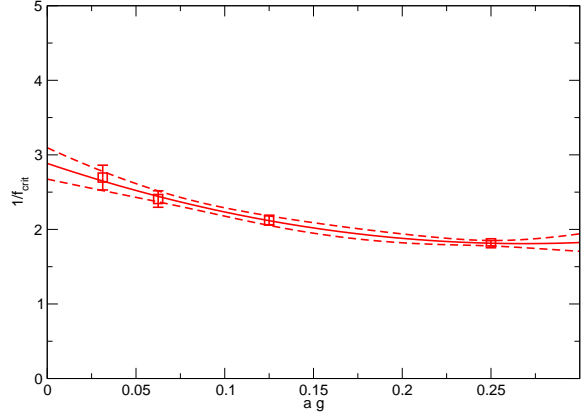


Figure 4: Continuum limit of the renormalised dimensionless critical coupling f_{crit} for the supersymmetry and $\mathbb{Z}(2)$ -symmetry breaking phase transition.